

ON THE UNRAMIFIED SPHERICAL AUTOMORPHIC SPECTRUM

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ABSTRACT. For a split connected reductive group G defined over a number field F , we compute the part of the spherical automorphic spectrum which is supported by the cuspidal data containing $(T, 1)$, where T is a maximal split torus and 1 is the trivial automorphic character. The proof uses the residue distributions which were introduced by the third author (in joint work with G. Heckman) in the study of graded affine Hecke algebras, and a result by M. Reeder on the weight spaces of the (anti)spherical discrete series representations of affine Hecke algebras. Note that both these ingredients are of a purely local nature. For many special cases of reductive groups G similar results have been established by various authors. The main feature of the present proof is the fact that it is uniform and general.

1. INTRODUCTION

Let G be a connected reductive group defined over a number field F . Denote by \mathbb{A} the ring of adèles of F . Assume for simplicity in this introductory part that G is semisimple. The space $L^2(G(F)\backslash G(\mathbb{A}))$ of square-integrable automorphic forms is a central object in the theory of automorphic forms and its relation to representation theory and number theory via the Langlands program.

Langlands [L] has given a decomposition of this space in terms of cuspidal data \mathfrak{X} , in which each $\mathfrak{X} := [M, \Xi]$ is an equivalence class of pairs (M, Ξ) with M an F -Levi subgroup of G and Ξ an orbit of certain character twists of a cuspidal automorphic representation of M :

$$(1) \quad L^2(G(F)\backslash G(\mathbb{A})) = \hat{\oplus}_{\mathfrak{X}} L^2(G(F)\backslash G(\mathbb{A}))_{\mathfrak{X}}.$$

More precisely, if a_M^* denotes the dual of the real Lie algebra of the maximal split torus A_M in the center of M , then its complexification $a_{M, \mathbb{C}}^*$ parametrizes in a natural way a certain class of “unramified” complex characters of M , and this implies a natural action of the complex vector space $a_{M, \mathbb{C}}^*$ on the set of cuspidal automorphic representations of M via character twist. The orbits Ξ in the statement above are the orbits for this action by $a_{M, \mathbb{C}}^*$ while (M, Ξ) is equivalent to (M', Ξ') if they are $G(F)$ -conjugate to each other. In other words, if σ is a representation in Ξ , the other elements of Ξ can be written in the form σ_λ , $\lambda \in a_{M, \mathbb{C}}^*$.

Letting act $G(\mathbb{A})$ on $L^2(G(F)\backslash G(\mathbb{A}))$ by right translations, the irreducible closed subspaces of $L^2(G(F)\backslash G(\mathbb{A}))$ are called discrete series representations. Among the

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discrete series representations are the cuspidal automorphic representations (those whose space contains a cuspidal automorphic form of $G(\mathbb{A})$, i.e. they appear in an $L^2(G(F)\backslash G(\mathbb{A}))_{\mathfrak{X}}$ with $\mathfrak{X} = [G, \Xi]$ and Ξ a singleton) and the residual spectrum (i.e. those discrete series representations which appear in an $L^2(G(F)\backslash G(\mathbb{A}))_{[M, \Xi]}$ with $M \neq G$). One of the main achievements of Langlands was to show that the representations in the residual spectrum correspond to certain residues of Eisenstein series via a complicate contour shift, which was the most difficult part in [L].

A discrete series representation π of $G(\mathbb{A})$ (and more generally any irreducible automorphic representation) admits a decomposition $\otimes'_v \pi_v$ into local components. If (M, Ξ) is the cuspidal datum corresponding to π , and σ is a base point of Ξ , then there is an element λ in the complex vector space $a_{M, \mathbb{C}}^*$, so that π is a subquotient of $I(\sigma_\lambda)$, the normalized induced representation to $G(\mathbb{A})$ from the character σ_λ of $P(\mathbb{A})$. For a given base point σ in Ξ , the contour shift procedure established by Langlands computes, among other things, those λ such that $I(\sigma_\lambda)$ has a discrete series subquotient. If a local component σ_v of σ in a finite place v is a supercuspidal representation, then it is a striking observation in known cases (cf. [K2, Conjecture 8.7] and [M2, Conjecture p. 817]) that the λ such that the automorphic representation $I(\sigma_\lambda)$ has a discrete series subquotient correspond to those λ for which the induced representation $I_v(\sigma_{\lambda, v})$ of the p -adic group $G(F_v)$ has a square-integrable subquotient. In [H1], using at a crucial step a trace argument inspired from [O1] (and in addition a nonvanishing result of the residues of the μ function therein), it has been shown that such λ in the context of p -adic groups correspond to residue points of Harish-Chandra's μ -function.

At first sight the contour shift procedure in the automorphic setting is quite different. The aim of this paper is to show that at least in the unramified spherical case the contour shift in the automorphic setting can be directly related to the contour shift in [H1] and [O1] (in fact, we will come nearer to the form in [HO1] in this unramified case). This enables us to prove the above observation uniformly in the spherical case for any split reductive group.

Now assume G is any connected reductive group, defined and split over F . For each place v of F , we let K_v be a maximal compact subgroup of G_v with, for all non-Archimedean places, $K_v = G(\mathfrak{o}_v)$, where \mathfrak{o}_v the ring of integers of F_v . Let $\mathbf{K} := \prod_v K_v$ be the corresponding maximal compact subgroup of $G(\mathbb{A})$. Fix an F -Borel subgroup $B = TU$ in G , where T is a maximal F -split torus in G and U is the unipotent radical of B . Denote by Z_G the group of adelic points of the center of G and consider the $G(\mathbb{A})$ -representation $L^2(G(F)Z_G\backslash G(\mathbb{A}))$ (which is isomorphic to the subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}))$ on which the center acts trivially).

Denote by $X^*(T)$ the lattice of rational characters of T and by $X_*(T)$ the dual lattice of cocharacters. Then a_T^* can be seen as the real vector space $\mathbb{R} \otimes X^*(T)$. Its dual is $a_T := \mathbb{R} \otimes X_*(T)$. We let $\Phi = \Phi(G, T) \subseteq a_T^*$ be the root system of G , Φ^+ the set of positive roots corresponding to B , Δ the set of simple roots and W the Weyl group of Φ . The real vector space spanned by the roots, which is orthogonal to $a_G := \mathbb{R} \otimes X_*(G) \subseteq a_T$, will be denoted $a_T^{G*} \subseteq a_T^*$ and its complexification will be denoted by $a_{T, \mathbb{C}}^{G*} \subseteq a_{T, \mathbb{C}}^*$.

In this paper, we are interested in spherical automorphic forms (i.e. the subrepresentation of $L^2(G(F)Z_G\backslash G(\mathbb{A}))$ generated by their \mathbf{K} -invariant functions) which are supported by the cuspidal datum $\mathfrak{X} = [T, 1]$, where 1 denotes the orbit of the

trivial character of $T(F)\backslash T(\mathbb{A})$. Let $L^2(G(F)Z_G\backslash G(\mathbb{A}))_{[T,1]}$ denote the space of automorphic forms supported by $[T, 1]$. Its space of \mathbf{K} -fixed vectors is topologically generated by the pseudo-Eisenstein series,

$$(2) \quad \theta_\phi(g) = \int_{\operatorname{Re}(\lambda)=\lambda_0 \gg 0} \phi(\lambda) E(\lambda, g) d\lambda,$$

in which $E(\lambda, g)$ denotes the unramified Borel Eisenstein series (see equation (55)), $\phi \in PW(a_{T,\mathbb{C}}^{G*})$, the space of Paley-Wiener function on $a_{T,\mathbb{C}}^{G*}$, and the notation $\operatorname{Re}(\lambda) = \lambda_0 \gg 0$ means that one integrates over λ with real part equal to a fixed element in the positive Weyl chamber of a_T^{G*} far away from the walls and the origin.

Remark 1.1. *More generally (see [MW2, Chapter II]), for each Paley-Wiener function ϕ with values in $\operatorname{ind}_{\mathbf{K} \cap T(\mathbb{A})}^{\mathbf{K}} 1$ one can define an Eisenstein series $E(\phi, \lambda)$ and a corresponding pseudo-Eisenstein series*

$$(3) \quad \theta_\phi(g) = \int_{\operatorname{Re}(\lambda)=\lambda_0 \gg 0} E(\phi, \lambda)(g) d\lambda.$$

The space $L^2(G(F)Z_G\backslash G(\mathbb{A}))_{[T,1]}$ is then generated by these pseudo-Eisenstein series. In the present case of \mathbf{K} -spherical functions, ϕ takes values in $(\operatorname{ind}_{\mathbf{K} \cap T(\mathbb{A})}^{\mathbf{K}} 1)^{\mathbf{K}} \simeq \mathbb{C}$ and one can check that the two expressions for the pseudo-Eisenstein series (2) and (3) agree.

If ϕ and ψ are in $PW(a_{T,\mathbb{C}}^{G*})$, then the formula for their inner product $(\theta_\phi, \theta_\psi)$ (cf. [MW2, II.2.1]) can be expressed in our setting in terms of the completed Dedekind zeta-function $\Lambda(s)$ (i.e. the zeta-function $\zeta(s)$ associated to F by Dedekind, completed at the infinite places and including the factor corresponding to the discriminant of F) in the following way, where $\phi^-(\lambda) := \overline{\phi(\bar{\lambda})}$ is again a Paley-Wiener function:

$$(4) \quad (\theta_\phi, \theta_\psi) = \int_{\operatorname{Re}(\lambda)=\lambda_0 \gg 0} \sum_{w \in W} \prod_{\alpha \in \Phi^+ \cap w^{-1}\Phi^-} \frac{\Lambda(\alpha^\vee(\lambda))}{\Lambda(\alpha^\vee(\lambda) + 1)} \phi^-(-w\lambda) \psi(\lambda) d\lambda.$$

Among others, $\Lambda(s)$ satisfies the well-known analytic properties:

- (a) $\Lambda(s)$ is meromorphic with simple poles at 0 and 1,
- (b) $\Lambda(s)$ has zeroes only for $0 < \operatorname{Re}(s) < 1$,
- (c) $\Lambda(s)$ satisfies a functional equation $\Lambda(s) = \Lambda(1 - s)$.

Our results will be formulated in terms of the L -group ${}^L G$ of G . As G is split, one can take here for ${}^L G$ the dual group of G , so that ${}^L G$ is a complex connected reductive group. From the celebrated results of Bala-Carter (see [BC] and [Car]), there is a bijection between the (finite) set of unipotent conjugacy classes \mathbf{o} of the dual group ${}^L G$ and the set of conjugacy class of pairs $\{({}^L M, {}^L P)\}$ with ${}^L M$ a Levi subgroup of ${}^L G$ and ${}^L P$ a distinguished parabolic subgroup of ${}^L M'$, the derived group of ${}^L M$. Fix, once and for all, a complete set of representatives for each ${}^L G$ -class with the assumptions that ${}^L M \subseteq {}^L G$ is a Levi which contains ${}^L T$, the maximal torus of ${}^L G$ corresponding to T , and ${}^L P$ is a parabolic subgroup of ${}^L M'$ containing ${}^L T \cap {}^L M'$. We have thus a bijection

$$\mathbf{o} \leftrightarrow ({}^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}}).$$

Each representative $({}^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}})$ uniquely defines a subspace $a_{M_{\mathbf{o}}}^* \subseteq a_T^*$, in which $M_{\mathbf{o}}$ is the Levi of G corresponding to ${}^L M_{\mathbf{o}}$. We will also write $a_{M_{\mathbf{o}}}^{G*}$ to denote the subspace of $a_{M_{\mathbf{o}}}^*$ which is orthogonal to a_G^* . Let $a_T^{M_{\mathbf{o}}*}$ be the subspace of a_T^* that is generated by the roots of $M_{\mathbf{o}}$ relative to T , so that $a_T^* = a_{M_{\mathbf{o}}}^* \oplus a_T^{M_{\mathbf{o}}*}$. Then, we also have a uniquely defined element $\lambda_{M_{\mathbf{o}}}(\mathbf{o}) \in a_T^{M_{\mathbf{o}}*}$ such that $2\lambda_{M_{\mathbf{o}}}(\mathbf{o})$ is the weighted Dynkin diagram of the distinguished parabolic. We will omit the subscript when $M_{\mathbf{o}} = G$.

Theorem 1. *For each unipotent orbit \mathbf{o} , there exists a positive measure $\mu_{\mathbf{o}}$ (see (50)) on the W -orbit of the affine subspace $L_{\mathbf{o}}^t := \lambda_{M_{\mathbf{o}}}(\mathbf{o}) + ia_{M_{\mathbf{o}}}^{G*}$ and a positive semidefinite Hermitian form $\langle \cdot, \cdot \rangle_{\mathbf{o}}$ on the space $PW(a_{T,\mathbb{C}}^{G*})$ such that*

$$(\theta_{\phi}, \theta_{\psi}) = \sum_{\mathbf{o}} \langle \phi, \psi \rangle_{\mathbf{o}},$$

for all $\phi, \psi \in PW(a_{T,\mathbb{C}}^{G*})$, which induces an isometry of Hilbert spaces

$$\begin{aligned} L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1]}^{\mathbf{K}} &\cong \left(\bigoplus_{\mathbf{o}} L^2(WL_{\mathbf{o}}^t, \mu_{\mathbf{o}}) \right)^W \\ &= \bigoplus_{\mathbf{o}} L^2(L_{\mathbf{o}}^t, \tilde{\mu}_{\mathbf{o}})^{W({}^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}})}, \end{aligned}$$

where $W({}^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}})$ is the Weyl group of $({}^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}})$, cf. (53) and $\tilde{\mu}_{\mathbf{o}} = m_{\mathbf{o}} \mu_{\mathbf{o}}|_{L_{\mathbf{o}}^t}$, with $m_{\mathbf{o}}$ the number of distinct affine subspaces of the form $w(L_{\mathbf{o}}^t)$ with $w \in W$.

Let $\mathcal{H}(G(\mathbb{A}), \mathbf{K}) = \otimes'_v \mathcal{H}(G_v, K_v)$ denote the global spherical Hecke algebra, in which $\mathcal{H}(G_v, K_v)$ denotes the corresponding spherical Hecke algebra of each local place v . This algebra is equipped with a $*$ -structure coming from each local factor (see (58) and (59)). It acts by convolution on the space of \mathbf{K} -invariant functions on $G(F)Z_G \backslash G(\mathbb{A})$ and diagonally on $\left(\bigoplus_{\mathbf{o}} L^2(WL_{\mathbf{o}}^t, d\mu_{\mathbf{o}}) \right)^W$ (see Section 3). Our main result can then be stated in terms of the holomorphic normalized Eisenstein series $E_0(\lambda, g)$ (see Section 3, (57)) as:

Theorem 2. *The transform $\mathcal{F} : L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1]}^{\mathbf{K}} \rightarrow \left(\bigoplus_{\mathbf{o}} L^2(WL_{\mathbf{o}}^t, \mu_{\mathbf{o}}) \right)^W$*

$$(5) \quad \mathcal{F}(f)(\lambda) = \int_{G(F)Z_G \backslash G(\mathbb{A})} f(g) E_0(-\lambda, g) dg$$

defined for $\lambda \in \cup_{\mathbf{o}} WL_{\mathbf{o}}^t$ is an isometry of $*$ -unitary $\mathcal{H}(G(\mathbb{A}), \mathbf{K})$ -modules.

Assuming that \mathbf{o} is distinguished, i.e., it does not intersect any proper Levi subgroup, for any Archimedean or non-Archimedean place v , denote by $\pi_{v,\lambda(\mathbf{o})}$ the unique irreducible spherical subquotient of the (unramified) principal series representation of the local group G_v induced by the character of T_v corresponding to $\lambda(\mathbf{o})$, and denote by $\pi_{\lambda(\mathbf{o})}$ the irreducible representation of $G(\mathbb{A})$ given by

$$(6) \quad \pi_{\lambda(\mathbf{o})} = \otimes'_v \pi_{v,\lambda(\mathbf{o})},$$

which has a \mathbf{K} -invariant vector. Each \mathbf{K} -invariant normalized Eisenstein series $E_0(\lambda(\mathbf{o}), g)$ (see Section 3, (57)) generates an admissible $G(\mathbb{A})$ -subrepresentation of $L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1]}$ ([HC], see also the survey [Co, Theorem 3.5]) whose space of \mathbf{K} -invariants is one dimensional. From the unitarity of $L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1]}$, it follows that this subrepresentation is irreducible and hence isomorphic to $\pi_{\lambda(\mathbf{o})}$. Moreover, these subrepresentations are inequivalent for distinct unipotent orbits \mathbf{o} .

Let $L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1],d}$ be the discrete part of $L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1]}$, i.e., the closure of the span of all topologically irreducible $G(\mathbb{A})$ -subrepresentations, we obtain the following representation-theoretic corollaries:

Corollary 1. *Let $L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1],d}^{\text{sph}}$ be the smallest closed and $G(\mathbb{A})$ -invariant subspace of $L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1],d}$ containing $(L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1],d})^{\mathbf{K}}$. This space is multiplicity-free and decomposes as*

$$L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1],d}^{\text{sph}} = \bigoplus_{\mathbf{o}} \pi_{\lambda(\mathbf{o})},$$

with the sum indexed by the finite set of distinguished unipotent orbits.

Proof. Let $\mathcal{A} := L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1]}$. Its subspace of \mathbf{K} -invariants is a module for the commutative C^* -algebra $C^*(G(\mathbb{A}), \mathbf{K})$, given as a C^* -completion of the convolution algebra $L^1(G(\mathbb{A}), \mathbf{K})$. Thus, there is a unique decomposition of the discrete part of $\mathcal{A}^{\mathbf{K}}$ which, by Theorem 2, is

$$(7) \quad (\mathcal{A}^{\mathbf{K}})_d \cong \bigoplus_{\mathbf{o}} \pi_{\lambda(\mathbf{o})}^{\mathbf{K}},$$

a finite multiplicity-free orthogonal direct sum of irreducible $C^*(G(\mathbb{A}), \mathbf{K})$ -modules. On the other hand, as $G(\mathbb{A})$ is of type I (see [Cl, Theorem A.1]), \mathcal{A}_d decomposes uniquely as an orthogonal direct sum of irreducible $G(\mathbb{A})$ -invariant subspaces. One checks that there is a bijection between irreducible $C^*(G(\mathbb{A}), \mathbf{K})$ -modules and irreducible representations of $G(\mathbb{A})$ with \mathbf{K} -fixed vector, by taking respectively the closure of the space generated by the global Hecke algebra and invariants by \mathbf{K} . From this and the fact that the space of \mathbf{K} -invariants of an irreducible $G(\mathbb{A})$ -subrepresentation of \mathcal{A} has positive measure [D, Proposition 8.6.8(ii)], one checks that $(\mathcal{A}_d)^{\mathbf{K}} = (\mathcal{A}^{\mathbf{K}})_d$. Comparing with (7), we obtain the result. \square

Corollary 2. *For each distinguished \mathbf{o} , the local factors $\pi_{v,\lambda(\mathbf{o})}$ of $\pi_{\lambda(\mathbf{o})}$ are unitary representations for all places v .*

Remark 1.2. *In many special classes of reductive groups similar results on the \mathbf{K} -spherical automorphic spectrum have been obtained [J], [L], [K1], [Mi], [M1] and [MW1]. (We remark that the results on exceptional groups in [Mi] are partly based on computer assisted computations.) We present here a new uniform and conceptual approach which also takes care of the remaining cases left open by the previous authors. Such results are all in accordance with Arthur's conjectures [A]. We hope that a similar, more elaborate, uniform method, based on the ideas presented here, can be applied to a more general setting.*

Let us give a rough outline of the main new argument in this paper. After some elementary manipulations we will recast the right hand side of (4) in a form using a functional $X_{V,pV}$ on the Paley-Wiener space (see (19) for the precise formula). This functional turns out to be intimately related to the functional defined in [HO1, Equation (3.8)] in the context of the harmonic analysis of graded affine Hecke algebras, but with the extra complication that the kernel is meromorphic rather than entire, with possible poles in a critical region derived from the critical zeroes of Λ . In the context of [HO1], it was shown that such a functional has a *canonical* decomposition as a finite sum of tempered residue distributions, which can be computed using a (not canonical) system of contour shifts. In the present context we meet the

additional challenge to show that one can choose a system of contour shifts which avoids the critical region at all times. This is nontrivial, since at first sight the contours defining the lower dimensional spectral series are lying in the critical region. However, based on an \mathfrak{sl}_2 -argument, whose essential part is Lemma 2.17, we shall prove that certain cancellations take place in the restriction of the kernel to the corresponding lower dimensional residual subspaces L , creating a window of holomorphy wide enough to perform the necessary contour shifts without ever seeing the critical poles (cf. Lemma 2.18). Also, using a suitable holomorphic normalization of the Eisenstein series (cf. Lemma 3.1) we write down an explicit integral transform to describe the automorphic spectrum.

2. THE SCALAR PRODUCT OF PSEUDO EISENSTEIN SERIES AND GRADED HECKE ALGEBRAS

2.1. The Hecke algebra c -function. We will denote by ${}^L\mathfrak{g}$ the Lie algebra of ${}^L G$. We will fix ${}^L\mathfrak{t} \subseteq {}^L\mathfrak{b} \subseteq {}^L\mathfrak{g}$ a Cartan and a Borel subalgebras corresponding to ${}^L T$ and ${}^L B$ and will denote the root system of $({}^L G, {}^L T)$ by Φ^\vee , which is the dual to Φ , the root system of (G, T) , and Δ^\vee is the set of simple roots. We have a decomposition

$$(8) \quad {}^L\mathfrak{g} = {}^L\bar{\mathfrak{n}} \oplus {}^L\mathfrak{t} \oplus {}^L\mathfrak{n},$$

with ${}^L\mathfrak{n}$ the nilpotent radical of ${}^L\mathfrak{b}$ and ${}^L\bar{\mathfrak{n}}$ its opposite. Moreover, we will work in a setting closely related to the one in [HO1] and use some results therein. Because of this we will adhere to those notations. We thus define

$$(9) \quad V := a_T^{G*},$$

where we recall that $a_T^{G*} \subseteq a_T^*$ is the subspace spanned by the roots of (G, T) . We will write $V_{\mathbb{C}}$ for the complexification of V . We fix once and for all a W -invariant Euclidean inner product on V , even though the results of this paper are independent of this choice.

With the identification (9), we have that $\Phi^\vee \subseteq V^*$, V_+ denote the fundamental chamber of V with respect to Δ^\vee and the pairing $V_{\mathbb{C}}^* \times V_{\mathbb{C}} \rightarrow \mathbb{C}$ is denoted by $(\xi, \lambda) \mapsto \xi(\lambda)$.

The measure $d\lambda$ of (2) is defined as follows. Let $\{y_1, \dots, y_r\}$ be a basis of the orthogonal projection of $X_*(T)$ onto V^* . Then $d\lambda$ on iV is given by

$$(10) \quad d\lambda := \frac{dy_1 \wedge \dots \wedge dy_r}{(-2\pi i)^r}.$$

Let us now rewrite (4) in such a way that the Residue Lemma [HO1, Lemma 3.1] becomes applicable. Put

$$(11) \quad \gamma_{\mathbb{H}}(s) := \frac{s+1}{s}, \quad \gamma_{\mathbb{H}}^-(s) := \gamma_{\mathbb{H}}(-s).$$

and write

$$(12) \quad c_{\mathbb{H}}(\lambda) := \prod_{\alpha \in \Phi^+} \gamma_{\mathbb{H}}(\alpha^\vee(\lambda)), \quad c_{\mathbb{H}}^-(\lambda) := c_{\mathbb{H}}(-\lambda),$$

so that one has $c_{\mathbb{H}}^-(\lambda) = \prod_{\alpha \in \Phi^+} \gamma_{\mathbb{H}}^-(\alpha^\vee(\lambda))$. We will also write $\gamma_{\mathbb{H}}^+$ and $c_{\mathbb{H}}^+$ for $\gamma_{\mathbb{H}}$ and $c_{\mathbb{H}}$ respectively, if this seems appropriate.

The rational function $c_{\mathbb{H}}$ on $V_{\mathbb{C}}$ is in fact the c -function of the graded Hecke algebra with (infinitesimal) Hecke parameter $k_\alpha = 1$ for all α (see [HO1, (1.8)]).

We define in addition a global c -function by

$$c(\lambda) := \prod_{\alpha \in \Phi^+} \gamma(\alpha^\vee(\lambda)),$$

with $\gamma(s) := \frac{1}{s^2 \Lambda(-s)}$. Define

$$(13) \quad \rho(s) := s(s-1)\Lambda(s).$$

Hence, we have

- (a) $\rho(s)$ is an entire function,
- (b) $\rho(s)$ has zeros only for $0 < \operatorname{Re}(s) < 1$,
- (c) $\rho(s)$ satisfies the functional equation $\rho(s) = \rho(1-s)$,
- (d) $\rho(s)$ is at most of polynomial growth as $|t| \rightarrow \infty$ on vertical lines $\sigma + it$,
- (e) $\rho(s)^{-1}$ is at most of polynomial growth on vertical lines $\sigma + it$ for $\sigma \geq 1$,

(the analytic properties of items (d) and (e) use the rapid decay of the gamma function on vertical lines and the estimates for $\zeta(s)$ and $\zeta(s)^{-1}$ near the line $\sigma = 1$, see e.g., [Stas] and [JL, Section 10.6]) and we see that

$$\gamma(s) = \frac{\gamma_{\mathbb{H}}(s)}{\rho(-s)} = \frac{\gamma_{\mathbb{H}}(s)}{\rho(s+1)}.$$

A straightforward computation shows that:

Lemma 2.1. *Let f be a function of one complex variable, and define a function φ on $V_{\mathbb{C}}$ by $\varphi(\lambda) = \prod_{\alpha \in \Phi^+} f(\alpha^\vee(\lambda))$. For all $w \in W$ we have*

$$\frac{\varphi(w\lambda)}{\varphi(\lambda)} = \prod_{\alpha \in \Phi^+ \cap w^{-1}\Phi^-} \frac{f(-\alpha^\vee(\lambda))}{f(\alpha^\vee(\lambda))}$$

Let r be the entire function on $V_{\mathbb{C}}$ defined by

$$r(\lambda) := \prod_{\alpha \in \Phi^+} \rho(\alpha^\vee(\lambda)).$$

As a consequence of Lemma 2.1 (and W -invariance of $c_{\mathbb{H}}(\lambda)c_{\mathbb{H}}(-\lambda)$ and $r(\lambda)r(-\lambda)$) we obtain various identities, like:

$$(14) \quad \begin{aligned} \frac{c(-w\lambda)}{c(-\lambda)} &= \prod_{\alpha \in \Phi^+ \cap w^{-1}\Phi^-} \frac{\Lambda(\alpha^\vee(\lambda))}{\Lambda(\alpha^\vee(\lambda) + 1)} \\ &= \frac{c_{\mathbb{H}}(-w\lambda)}{c_{\mathbb{H}}(-\lambda)} \frac{r(\lambda)}{r(w\lambda)} = \frac{c_{\mathbb{H}}(\lambda)}{c_{\mathbb{H}}(w\lambda)} \frac{r(\lambda)}{r(w\lambda)} = \frac{c(\lambda)}{c(w\lambda)}. \end{aligned}$$

Similarly, we see that

$$(15) \quad \frac{r(\lambda)}{r(w\lambda)} = \prod_{\alpha \in \Phi^+ \cap w^{-1}\Phi^-} \frac{\rho(\alpha^\vee(\lambda))}{\rho(\alpha^\vee(\lambda) + 1)}.$$

It follows that the pole set of the meromorphic function $\frac{r(\lambda)}{r(w\lambda)}$ is a union of hyperplanes of the form $\alpha^\vee(\lambda) = z$ where $-1 < \operatorname{Re}(z) < 0$ and $\alpha \in \Phi^+ \cap w^{-1}\Phi^-$.

Inserting (14) in (4) we obtain

$$(16) \quad (\theta_\phi, \theta_\psi) = \int_{\operatorname{Re}(\lambda) = \lambda_0 \gg 0} R_\phi(\lambda) \psi(\lambda) \frac{d\lambda}{c_{\mathbb{H}}(-\lambda)},$$

in which the meromorphic function R_ϕ is given by

$$(17) \quad R_\phi(\lambda) := \sum_{w \in W} c_{\mathbb{H}}(-w\lambda) \phi^-(-w\lambda) \frac{r(\lambda)}{r(w\lambda)}.$$

We note in passing that the meromorphic function $r^{-1}R_\phi$ is W -invariant. Due to its importance, we define the summand $R_{\phi,w}$ as the rational function of $V_{\mathbb{C}}$

$$(18) \quad R_{\phi,w}(\lambda) := c_{\mathbb{H}}(-w\lambda) \phi^-(-w\lambda) \frac{r(\lambda)}{r(w\lambda)},$$

so that we can write $R_\phi = \sum_w R_{\phi,w}$.

For a point $p_V \in V$ outside the set of poles of $c_{\mathbb{H}}(-\lambda)^{-1}$ we define a linear functional X_{V,p_V} on $PW(V_{\mathbb{C}})$, the space of Paley-Wiener functions on $V_{\mathbb{C}}$, by (cf. [HO1, equation (3.8)]):

$$(19) \quad X_{V,p_V}(f) := \int_{\operatorname{Re}(\lambda)=p_V} f(\lambda) \frac{d\lambda}{c_{\mathbb{H}}(-\lambda)}.$$

For $p_V = \lambda_0$ as in (2), we use (19) to rewrite (16) as

$$(20) \quad (\theta_\phi, \theta_\psi) = X_{V,p_V}(\psi R_\phi).$$

We would like to apply the support results [HO1, Proposition 3.6, Corollary 3.7, Theorem 3.13] and [O2, Theorem 6.1] to the expression (20). In order to do so we need to show that the additional poles of R_ϕ coming from the factors $\frac{r(\lambda)}{r(w\lambda)}$ with $w \in W$ do not interfere with the process of iteratively taking residues as in [HO1].

2.2. Residual distributions. Given an affine subspace $L \subseteq V$, write $L = c_L + V^L$, where $V^L \subseteq V$ is a subspace, and c_L is the element of L which is of minimum distance to the origin. We call c_L the **center of L** , and we write $L_{\mathbb{C}} = c_L + V_{\mathbb{C}}^L \subseteq V_{\mathbb{C}}$ for its complexification. We define the **tempered form of L** to be $L^t := c_L + iV^L$. Given $\alpha^\vee \in V^*$, we will write $\alpha^\vee(L) = \text{cst}$ to indicate that α^\vee is constant on L . An affine subspace of V defines a subset $\Phi_L \subseteq \Phi$ consisting of all roots $\alpha \in \Phi$ whose coroots are constant on L . We will denote by $V_L \subseteq V$ the span of Φ_L .

Definition 2.2. *An affine subspace $L \subseteq V$ is called a **residual subspace** if*

$$(21) \quad |\{\alpha \in \Phi \mid \alpha^\vee(L) = 1\}| = |\{\alpha \in \Phi \mid \alpha^\vee(L) = 0\}| + \operatorname{codim}_V(L).$$

We will denote the set of all residual subspaces by \mathcal{L} and the set of centers of residual subspaces by \mathcal{C} .

Remark 2.3. *Note that V is trivially residual. Further, the notion of residual subspace depends on (V, Φ) . In particular, the definition above implies that, if $L = c_L + V^L$ is a residual subspace of V , then c_L is a residual point with respect to (V_L, Φ_L) .*

Write $\omega(\lambda) := \frac{d\lambda}{c_{\mathbb{H}}(-\lambda)}$ for the rational n -form of the functional X_{V,p_V} defined above in (19), and denote by $\mathcal{H}(\omega)$ the collection of hyperplanes it defines. Let $\mathcal{L}(\omega)$ be the lattice of intersection of elements of $\mathcal{H}(\omega)$ and $\mathcal{C}(\omega)$ be the set of their centers. Using the Residue Lemma [HO1, Lemma 3.1], it follows that there is a unique collection of tempered distributions $\{X_c \mid c \in \mathcal{C}(\omega)\}$ such that

- (a) $\operatorname{supp}(X_c) \subseteq \cup \{iV^L \mid L \in \mathcal{L}(\omega) \text{ with } c_L = c\}$
- (b) $X_{V,p_V}(f) = \sum_{c \in \mathcal{C}(\omega)} X_c(f(c + \cdot))$, for all $f \in PW(V_{\mathbb{C}})$.

Elements in the collection $\{X_c \mid c \in \mathcal{C}(\omega)\}$ just described, are called **local contributions**. Among them, many are identically zero. A first step towards determining which X_c is nonzero, and thus the support of X_{V,p_V} , is to introduce the functional $Y_{V,p_V} \in PW(V_{\mathbb{C}})^*$ given by

$$(22) \quad Y_{V,p_V}(f) := \int_{\operatorname{Re}(\lambda)=p_V} f(\lambda) \eta(\lambda),$$

for all $f \in PW(V_{\mathbb{C}})$ and with $\eta(\lambda) := \frac{d\lambda}{c_{\mathbb{H}}(-\lambda)c_{\mathbb{H}}(\lambda)}$. We similarly define the sets $\mathcal{H}(\eta)$, $\mathcal{L}(\eta)$ and $\mathcal{C}(\eta)$. We then apply the Residue Lemma to Y_{V,p_V} to obtain a collection of local distributions $\{Y_c \mid c \in \mathcal{C}(\eta)\}$ whose properties we summarize in the following Theorem. In it, for each residual subspace $L \in \mathcal{L}$, from $V = V_L \oplus V^L$, we let p_L denote the projection of p_V onto V_L , $r_L := \dim V^L$ and $\{y_1, \dots, y_{r_L}\}$ be a basis a basis of the canonical projection of the lattice $X_*(T) \cap (V_L)^{\perp}$ onto V^* . We have a lower rank functional Y_{V_L,p_L} and we denote by Y_{Φ_L,c_L} its local contribution at the (V_L, Φ_L) -distinguished point $c_L \in V_L$.

Theorem 2.4. *For $c \in \mathcal{C}(\eta) \cap \overline{V_+}$, the local contribution Y_c of Y_{V,p_V} admits a decomposition*

$$(23) \quad Y_c(f(c + \cdot)) = \sum_{\{L \in \mathcal{L}(\eta) \mid c_L = c\}} Y_L(f(c + \cdot)),$$

where the functional Y_L is a positive measure on iV with support on iV^L that satisfies $Y_L = 0$ unless L is residual. More precisely, Y_L is defined by

$$Y_L(g) := Y_{\Phi_L,c_L}(\{c_L\}) \int_{iV^L} g(\lambda^L) d\mu^L(\lambda^L),$$

for any $g \in PW(V_{\mathbb{C}})$, in which, $Y_{\Phi_L,c_L}(\{c_L\})$ denotes the total mass of Y_{Φ_L,c_L} and for $\lambda^L \in iV^L$,

$$(24) \quad d\mu^L(\lambda^L) := \prod_{\alpha \in \Phi^+ \setminus \Phi_L} \frac{\alpha^{\vee}(c_L)^2 + \alpha^{\vee}(\operatorname{Im} \lambda^L)^2}{(\alpha^{\vee}(c_L) - 1)^2 + \alpha^{\vee}(\operatorname{Im} \lambda^L)^2} d\lambda^L$$

with $d\lambda^L$ a measure on iV^L given by

$$(25) \quad d\lambda^L := \frac{dy_1 \wedge \dots \wedge dy_{r_L}}{(-2\pi i)^{r_L}}.$$

Moreover, in each W -orbit of residual subspaces there exists at least one element L for which $Y_{\Phi_L,c_L}(\{c_L\}) > 0$.

Proof. The first part is similar to [HO1, Theorem 3.13]. The moreover part is item (A) of [O2, Theorem 6.1]. \square

Further, there is a precise link between the local contributions of X_{V,p_V} and Y_{V,p_V} which we describe in what follows. But for that, given any $c \in V$, let W_c denote the isotropy group of c and define the **averaging operator** A_c acting on meromorphic functions on $V_{\mathbb{C}}$ via

$$(26) \quad A_c(f)(\lambda) := \frac{1}{|W_c|} \sum_{v \in W_c} c_{\mathbb{H}}(v\lambda) f(v\lambda).$$

These operators were introduced in [HO1, (3.10)] and satisfy the property that, if f is holomorphic on a tubular neighborhood $U + iV$ of $c + iV$, then $A_c(f)$ is also

holomorphic. Our interest is when $c \in \mathcal{C}(\eta)$. For example, when $c = 0$, the center of the (trivially) residual subspace V and for the exponential function e^ξ defined as $\lambda \mapsto e^{\xi(\lambda)}$, for $\xi \in V$, we obtain

$$(27) \quad A_0(e^\xi)(\lambda) = \frac{1}{|W|} \sum_{w \in W} c_{\mathbb{H}}(w\lambda) e^{\xi(w\lambda)},$$

which is the elementary spherical function $\phi(\lambda, 1, \xi)$ discussed in [HO1, (1.9)] with parameter $k_\alpha = 1$, for all α . We will also use the version

$$A_c^-(f)(\lambda) = \frac{1}{|W_c|} \sum_{w \in W_c} c_{\mathbb{H}}^-(w\lambda) f(w\lambda),$$

corresponding to the parameter $k_\alpha = -1$ for all α (see (12)). We now state the promised link between the local contributions of X_{V,p_V} and Y_{V,p_V} . It is a slight variation of [HO1, Proposition 3.6].

Proposition 2.5. *Let $c \in \mathcal{C}(\eta) \cap \overline{V_+}$. Then for all $w \in W$ we have:*

$$(28) \quad X_{wc} = Y_c \circ w^{-1} \circ A_{wc}.$$

Remark 2.6. *We can generalize the definition of the averaging operators A_c as in (26) to $A_{\Psi,c}$ for any root system $\Psi \subseteq V$, independently if Ψ spans V or not. For that, let $W(\Psi)$ be its Weyl group, $W(\Psi)_c$ the stabilizer of c in $W(\Psi)$ and $c_{\mathbb{H},\Psi}$ be the c -function $c_{\mathbb{H}}$ with only roots of Ψ^+ as factors. We will omit the reference to Ψ when $\Psi = \Phi$.*

Corollary 2.7. *A local contribution X_c of the functional X_{V,p_V} satisfies $X_c = 0$, unless $c = c_L$ with L residual. In particular,*

$$(29) \quad X_{V,p_V}(f) = \sum_{c \in \mathcal{C}} X_c(f(c + \cdot)),$$

for all $f \in PW(V_{\mathbb{C}})$.

Proof. Note that $\mathcal{C}(\omega) \cap \overline{V_+} = \mathcal{C}(\eta) \cap \overline{V_+}$ and that $\mathcal{C}(\omega) \subseteq W(\mathcal{C}(\omega) \cap \overline{V_+})$. The result now follows from Proposition 2.5 and Theorem 2.4. \square

Looking closer to the existence part of the Residue Lemma one obtains, similarly to (23), that there is a decomposition of each local contribution of X_{V,p_V} as

$$(30) \quad X_c(f(c + \cdot)) = \sum_{\{L \in \mathcal{L} \mid c_L = c\}} X_L(f(c + \cdot)),$$

for some distributions X_L on $c + iV$ with support on L^t . Such a discussion was carried out in [O2, Section 3.1, (3.28)] using results of [O1]. The splitting of X_c in terms of a collection of distributions $\{X_L \mid L \text{ such that } c_L = c\}$ as in (30) is not canonical, as it depends on how one deforms the contours of integration. But the sum X_c is canonical. Our strategy to compute (4) will be to show that we can construct a collection of distributions by an iterative shift of contours in such a way that we never cross the pole-set of (17).

Remark 2.8. *A consequence of the definition of X_L as in [O2] is that, if nonzero its support is always L^t . In particular, X_c is nonzero if and only if there is an L with $c_L = c$ and X_L is nonzero.*

Remark 2.9. *The distributions X_L , which a priori are distributions on $PW(V_{\mathbb{C}})$, are extended to functions involving factors $\rho(\alpha^{\vee}(\lambda))$ or $\rho(\alpha^{\vee}(\lambda)+1)^{-1}$, for $\lambda \in L^t = c_L + iV^L$, as long as $\alpha^{\vee}(c_L) \geq 0$, because of the polynomial growth of these factors in this region (see properties (d) and (e) of $\rho(s)$ in Section 2.1).*

2.3. Vanishing conditions. The aim of this subsection is to give a sharp criterion to determine when the local contributions $\{X_c \mid c \in \mathcal{C}\}$ of (29) are nonzero. The choices of ${}^L B$ and ${}^L T$ made in Section 2.1 yield a based root datum $\mathcal{R}^{\vee} = (X_*(T), \Phi^{\vee}, X^*(T), \Phi, \Delta^{\vee})$. We let $q \in \mathbb{R}_{>1}$ be the cardinality of the residue field of the completion of F at a non-Archimedean place. Consider $\mathcal{H}(\mathcal{R}^{\vee}, q)$, the corresponding affine Hecke algebra, as in [O1, Theorem 2.2] and we let $\mathbb{H}(\mathcal{R}^{\vee}, 1)$ denote the associated graded algebra with equal parameter 1. The idea is to link the distributions X_{wc} , with the representation theory of these algebras. For that, we will need some preliminary facts. We start with the relation between affine residual subspaces and nilpotent orbits of ${}^L \mathfrak{g}$. The following proposition summarizes what we need, and proofs can be found in [O1, Appendix B] (see also [O2, Section 7] and [H2, Proposition 6.2]).

Proposition 2.10. (a) *There is a bijective correspondence $\mathfrak{o}_{WL} \leftrightarrow WL$ between the set of nilpotent orbits of the Langlands dual Lie algebra ${}^L \mathfrak{g}$ and the set of W -orbits of residual subspaces.*
 (b) *There is a bijective correspondence between the set of W -orbits of residual subspaces and the set of W -orbits of centers $c = c_L$ of residual subspaces. In particular, distinct W -orbits of tempered residual subspaces are disjoint.*
 (c) *For each residual L with center c_L there exists a Lie algebra homomorphism $\varphi : \mathfrak{sl}_2 \rightarrow {}^L \mathfrak{g}$ with $\varphi(e) = n \in \mathfrak{o}_{WL}$ and $\varphi(h) = 2c_L$, with $\{e, h, f\}$ the standard basis of \mathfrak{sl}_2 .*
 (d) *If $c = c_L$ then $\alpha^{\vee}(2c_L) \in \mathbb{Z}$ for all $\alpha \in \Phi$, and $\alpha^{\vee}(c_L) \in \mathbb{Z}$ for all $\alpha \in \Phi_L$.*
 (e) *If we choose $c = c_L$ in its Weyl group orbit so that c is dominant, then $2c$ is the weighted Dynkin diagram associated to \mathfrak{o}_{WL} .*

Recall that a residual subspace L defines a parabolic root system Φ_L . We let $W(\Phi_L)$ denote the Weyl group of this root system and we define $\text{Fix}(L) \subseteq W(\Phi_L)$ to be the pointwise fixator of L . Let also

$$(31) \quad c_{\mathbb{H},L}(\lambda) := \prod_{\alpha \in \Phi_L^+} \gamma_{\mathbb{H}}(\alpha^{\vee}(\lambda))$$

and we define $c_{\mathbb{H}}^L$ by the equation $c_{\mathbb{H}}(\lambda) = c_{\mathbb{H}}^L(\lambda)c_{\mathbb{H},L}(\lambda)$. Let \mathcal{M}^L denote the localization of the ring $\mathcal{O}(V_{\mathbb{C}})$ of holomorphic functions on $V_{\mathbb{C}}$ with respect to the complement of the prime ideal corresponding to $L_{\mathbb{C}}$. Let also $\mathcal{M}(L_{\mathbb{C}})$ denote the ring of meromorphic functions on $L_{\mathbb{C}}$. There is a well-defined evaluation map $e_L : \mathcal{M}^L \rightarrow \mathcal{M}(L_{\mathbb{C}})$ restricting an element of \mathcal{M}^L to $L_{\mathbb{C}}$. Observe that $c_{\mathbb{H}}^L \in \mathcal{M}^L$ according to the above condition. We define an operator $A^L \in \text{End}(\mathcal{M}^L)$ by

$$(32) \quad A^L(f)(\lambda) := |\text{Fix}(L)|^{-1} \sum_{w \in \text{Fix}(L)} c_{\mathbb{H}}(w\lambda)f(w\lambda).$$

Remark 2.11. *Observe that $\mathcal{O}(V_{\mathbb{C}})$ is not mapped to itself because of the factors $c_{\mathbb{H}}^L \in \mathcal{M}^L$. Note also that $A^L(f) = A_{\Phi_L, c_L}(c_{\mathbb{H}}^L f)$. In particular $A^L = A_c$ if $L = \{c\}$,*

a residual point. More generally, if $c = c_L$, we also have:

$$(33) \quad A_c(f)(\lambda) = \frac{|\text{Fix}(L)|}{|W_c|} \sum_{w \in W_c / \text{Fix}(L)} A^{wL}(f)(w\lambda).$$

Finally, these operators satisfy:

$$(34) \quad A^L(f)(\lambda) = A^{L,-}(f \circ (-id))(-\lambda).$$

Lemma 2.12. *Let L be a residual subspace with $c_L = c \in \mathcal{C} \cap \overline{V_+}$ and $w \in W$. Then, for every $f \in PW(V_{\mathbb{C}})$, it holds that*

$$X_{wc}(f(wc + \cdot)) = Y_{\Phi_L, c}(\{c\}) \sum_{\{M \in \mathcal{L} \mid c_M = c\}} \int_{M_t} A^{wM}(f)(w\lambda) d\mu^L(\text{Im}(\lambda)).$$

Proof. We use (28), (33) and Theorem 2.4 to obtain that, for all Paley-Wiener function f , it holds that

$$\begin{aligned} X_{wc}(f(c + \cdot)) &= \sum_{\{L \mid c_L = c\}} Y_L(A_{wc}(f)(w(c + \cdot))) \\ &= Y_{\Phi_L, c}(\{c\}) \sum_{\{M \mid c_M = c\}} \int_{M_t} A^{wM}(f)(w\lambda) d\mu^L(\text{Im}(\lambda)), \end{aligned}$$

where, for the last equality, we used that $Y_{\Phi_L, c}(\{c\}) = Y_{\Psi_M, c}(\{c\})$, for all L, M such that $c_L = c = c_M$ (see [O1, Theorem 3.27(i)]). \square

In [Re1, Corollary 5.13], Reeder gave a condition for the nonvanishing of the weight spaces of the unique anti-spherical constituent of the unramified minimal principal series for the Hecke algebra $\mathcal{H}(\mathcal{R}^\vee, q)$, with central character $W\tau$, for $\tau \in V$. More precisely, let $\mathcal{U}_-(\tau)$ denote the unique anti-spherical module as above and assume that \mathfrak{q}' , the q^{-1} -eigenspace of the action of $\text{Ad}(\tau)$ on ${}^L\mathfrak{g}$ is contained in ${}^L\mathfrak{n}$. We have:

Theorem 2.13 ([Re1]). *If $w \in W$, the weight space $\mathcal{U}_-(\tau, w\tau) \neq 0$ if and only if*

$$\mathfrak{q}'_0 \cap \text{Ad}(w^{-1})({}^L\mathfrak{n}) \neq \emptyset,$$

where \mathfrak{q}'_0 is the unique open orbit of $M = C_{LG}(\tau)^\circ$ on \mathfrak{q}' .

Using Reeder's result, we will obtain a precise understanding of the support of the functional $X_{V, pV}$. Let $\mathfrak{q} := \{x \in {}^L\mathfrak{g} \mid \text{ad}(L)x = x\}$ and $C := C_{LG}(L)^\circ$. Also, let \mathfrak{q}_0 denote the unique dense orbit of C on \mathfrak{q} .

Proposition 2.14. *Let L be a residual coset and let $c = c_L$ be its center. Assume that the residual subspace L is chosen in its W -orbit such that $c = c_L$ is dominant. Let $\mathfrak{o}_{WL} \subseteq {}^L\mathfrak{g}$ be the nilpotent orbit associated with WL as in Proposition 2.10 and let $w \in W$. The following are equivalent:*

- (a) *There exists a homomorphism $\varphi : \mathfrak{sl}_2 \rightarrow {}^L\mathfrak{g}$ such that $\varphi(h) = 2c$ and $\varphi(e) \in \mathfrak{q} \cap \mathfrak{o}_{WL} \cap \text{Ad}(w^{-1})({}^L\mathfrak{n})$.*
- (b) *The restriction to wL of $A^{wL}(f)$ is not identically equal to 0 for some Paley-Wiener functions f on $V_{\mathbb{C}}$.*
- (c) *The distribution X_{wc} is nonzero.*

Proof. It will be proven (a) \Leftrightarrow (b) \Leftrightarrow (c). For the first, note that $\mathfrak{q}_0 = \mathfrak{o}_{WL} \cap \mathfrak{q}$. Pick $\lambda \in V^L$ such that $c + i\lambda \in L_{\mathbb{C}}$ is a generic element and let $t := q^{i\lambda}$. We are then in condition to apply [Re1, Corollary 5.13] and conclude that (a) is equivalent to the nonvanishing of the $w\tau$ weight space of the unique antispherical module for $\mathcal{H}(\mathcal{R}^\vee, q)$ with central character $W\tau$, for $\tau = q^{-c}t^{-1}$. Using the involution $\iota : \mathcal{H}(\mathcal{R}^\vee, q^{-1}) \rightarrow \mathcal{H}(\mathcal{R}^\vee, q)$ that sends $T_i \mapsto -q^{-1}T_i$ (cf. [HO2, (3.8)]) and Lusztig's reduction theorems [Lu] we see that the last assertion is equivalent to the nonvanishing of the $-w(c + i\lambda)$ -weight space of the unique spherical module for $\mathbb{H}(\mathcal{R}^\vee, -1)$ with central character $-W(c + i\lambda)$. To prove that this last condition is equivalent to (b), let $\mu := c + i\lambda$. From [HO1, Section 2], the unique spherical module for $\mathbb{H}(\mathcal{R}^\vee, -1)$ with central character $W(-\mu)$ is generated by the elementary spherical function (cf. [HO1, (1.9)] and (27)), which can be written as

$$A_0^-(e^\xi)(-\mu) = \frac{|\text{Fix}(L)|}{|W|} \sum_{w \in W/\text{Fix}(L)} A^{wL, -}(e^\xi)(-w\mu),$$

for all $\xi \in V_{\mathbb{C}}^*$. Thus, using (34), the nonvanishing of the weight space is equivalent to saying that $A^{wL}(e^{-\xi})(w\mu)$ is not constantly zero as a function of ξ . Let us now show that, if $A^{wL}e^{-\xi}(w\mu) = 0$ for all $\mu \in L_{\mathbb{C}}$, then it holds that $A^{wL}f|_{wL} \equiv 0$ for all $f \in PW(V_{\mathbb{C}})$. From the Euclidean Paley-Wiener Theorem, given $f \in PW(V_{\mathbb{C}})$, there is a compactly supported smooth function $\varphi \in C_c^\infty(V^*)$ such that $f(\lambda) = \int_{\xi \in V^*} \varphi(\xi) e^{-\xi(\lambda)} d\xi$. Thus, for all $\lambda \in V_{\mathbb{C}}$

$$(A^{wL}f)(w\lambda) = \int_{\xi \in V^*} \varphi(\xi) (A^{wL}e^{-\xi})(w\lambda) d\xi,$$

in which the exchange of the summation symbol and the integration symbol is justified because of the compact support of φ . It follows from this that the vanishing of $(A^{wL}e^{-\xi})(w\mu)$, as a function of ξ , for $\mu \in L_{\mathbb{C}}$, implies $(A^{wL}f)|_{wL} = 0$, for all Paley-Wiener functions in $V_{\mathbb{C}}$. Conversely, suppose $(A^{wL}e^{-\xi})(w\mu)$ is nonvanishing. Let f_1 be a W_{wc} -invariant Paley-Wiener function on $V_{\mathbb{C}}$ such that $f_1(w\mu) \neq 0$. Put $f_2 = f_1 e^{-\xi} \in PW(V_{\mathbb{C}})$. Then,

$$(A^{wL}f_2)(w\mu) = f_1(w\mu) (A^{wL}e^{-\xi})(w\mu) \neq 0.$$

We thus proved (a) is equivalent to (b).

In view of Lemma 2.12, (c) implies (b). Conversely, note that each residual subspace $L_{\mathbb{C}}$ is determined by a finite set of affine hyperplanes $\alpha^\vee(\lambda) = 1$. For each such hyperplane, let $g_\alpha(\lambda) := (\alpha^\vee(\lambda) - 1)$ be the corresponding defining function. Then,

$$(35) \quad g(\lambda) = \prod g_\alpha,$$

with the product over all $\alpha \in \Phi$ such that $\alpha^\vee(L)$ is not constantly 0, is a holomorphic $\text{Fix}(L)$ -invariant function on $V_{\mathbb{C}}$ such that $g|_M$ is constantly 0 for all $M \neq L$. Assuming (b) and applying Lemma 2.12 for the function ${}^wgf \in PW(V_{\mathbb{C}})$, (with ${}^w g(\lambda) = g(w^{-1}\lambda)$), one has

$$X_{wc}({}^wgf(wc + \cdot)) = Y_{\Phi_L, c}(\{c\}) \int_{L^t} g(\lambda) A^{wL}(f)(w\lambda) d\mu^L(\text{Im}(\lambda)).$$

and the result will follow if $Y_{\Phi_L, c}(\{c\})$ is nonzero. If it were the case that $Y_{\Phi_L, c}(\{c\})$ is zero, then the unique spherical representation with central character $W(\Phi_L)c$ of

the lower rank graded Hecke algebra $\mathbb{H}_L(\Phi_L^{\vee,+}, -1)$ would not be a discrete series. But, assumption (b) implies that this module is a discrete series, using Reeder's result [Re1, Corollary 5.13] and [KL]. \square

At this point, we have enough information about the functional X_{V,p_V} at our disposal, and we remind the reader that our goal is to evaluate (20). Our next task will be to determine when the summand $R_{\phi,w}|_{L_{\mathbb{C}}}$ of (18) is nonvanishing, for $L \in \mathcal{L}$.

Lemma 2.15. *For any $w \in W$ and any residual L we have $r(-w\lambda)^{-1} \in \mathcal{M}^L$.*

Proof. The set of poles of $r(-w\lambda)^{-1}$ is a union of hyperplanes of the form $\alpha^{\vee}(\lambda) = z$ with $\operatorname{Re}(z) \in (-1, 0)$. On the other hand, if α^{\vee} is constant on L then $\alpha \in \Phi_L$, and by Proposition 2.10(c) this implies that $\alpha^{\vee}(L) \in \mathbb{Z}$. \square

By the previous lemma we conclude that it makes sense to write

$$R_{\phi}(-\lambda) = r(-\lambda)|W|A_0 \left(\frac{\phi^-}{r \circ (-id)} \right) (\lambda)$$

and that the argument of the operator A_0 belongs to \mathcal{M}^L for every residual subspace L . Let L be a residual subspace, with pointwise fixator $\operatorname{Fix}(L) \subseteq W$. Let $W^L \subseteq W$ denote a complete set of representatives for $W/\operatorname{Fix}(L)$. We can write the last equation, using (33), in the form

$$R_{\phi}(-\lambda) = r(-\lambda)|\operatorname{Fix}(L)| \sum_{u \in W^L} \left(A^{uL} \left(\frac{\phi^-}{r \circ (-id)} \right) \circ u \right) (\lambda),$$

or equivalently, using (34)

$$(36) \quad R_{\phi}(\lambda) = r(\lambda)|\operatorname{Fix}(L)| \sum_{u \in W^L} \left(A^{uL,-} \left(\frac{\phi^- \circ (-id)}{r} \right) \circ u \right) (\lambda).$$

In particular, the restriction of R_{ϕ} to $L_{\mathbb{C}}$ is well defined as a meromorphic function. We end this subsection with the promised condition about the vanishing of $R_{\phi,w}|_{L_{\mathbb{C}}}$.

Corollary 2.16. *Let L be as in Proposition 2.14 and let $v, w \in W$. The only summands $R_{\phi,w}$ of (18) which have a nonzero restriction to $vL_{\mathbb{C}}$ are those such that there is an \mathfrak{sl}_2 -homomorphism with $\varphi(h) = 2c$ and $\varphi(e) \in \mathfrak{q}_0 \cap \operatorname{Ad}(v^{-1}w^{-1})({}^L\mathfrak{n})$. This set of Weyl group elements is a union of left cosets for W_{vL} .*

Proof. If $u \in \operatorname{Fix}(L)$ then $\operatorname{Ad}(u)$ comes from conjugation by an element of $N_C({}^L\mathfrak{t})$. Therefore $\operatorname{Ad}(u)(\mathfrak{q}_0) = \mathfrak{q}_0$, showing that the condition on w is preserved by multiplication on the right with elements of W_{vL} .

Next, we analyze the restriction to vL of the summands $R_{\phi,w}$ of R_{ϕ} . Since for every residual subspace L one has $-L \in WL$ (see [HO1], or [O2] for an intrinsic argument), let $\mu = -w_0\lambda \in L$, with $\lambda \in L$ and w_0 the longest element of W . Using (36), (18) and (34) we get

$$R_{\phi,w}(v\lambda) = \left(r \left[A^{wvw_0L} \left(\frac{\phi^-}{r \circ (-id)} \right) \circ (-w) \right] \right) (v\lambda),$$

thus, it holds that $R_{\phi,w}|_{vL_{\mathbb{C}}}$ is nonzero if and only if $A^{wvw_0L} \left(\frac{\phi^-}{r \circ (-id)} \right)$ restricted to $wvw_0L_{\mathbb{C}}$ is nonzero, which implies that there exists a Paley-Wiener function f such that $A^{wvw_0L}(f)$ is not constantly zero. From Proposition 2.14, we can find φ such that $\varphi(e) \in \mathfrak{q}_0 \cap \operatorname{Ad}(w_0v^{-1}w^{-1})({}^L\mathfrak{n})$. Replacing φ by $\varphi' := \gamma \circ \operatorname{Ad}(w_0) \circ \varphi$ where γ is

the Cartan involution of ${}^L\mathfrak{g}$, one checks that φ' satisfies the criterion. The converse is similar. \square

2.4. The contour shift argument. Let $L \in \mathcal{L}$ be such that c_L is dominant, $v, w \in W$ and assume that both $X_{vc_L} \neq 0$ and that $R_{\phi, w}|_{vL_{\mathbb{C}}} \neq 0$. (Remark that these conditions are never satisfied if $w = 1$). Using Proposition 2.14 and Corollary 2.16, there exist Lie algebra homomorphisms $\varphi', \varphi'' : \mathfrak{sl}_2 \rightarrow {}^L\mathfrak{g}$ such that $\varphi'(h) = \varphi''(h) = 2c_L$ and

$$(37) \quad \begin{aligned} \varphi'(e) &\in \mathfrak{q}_0 \cap \text{Ad}(v^{-1})({}^L\mathfrak{n}) \\ \varphi''(e) &\in \mathfrak{q}_0 \cap \text{Ad}(v^{-1}w^{-1})({}^L\bar{\mathfrak{n}}) \end{aligned}$$

As both $\varphi'(e), \varphi''(e) \in \mathfrak{q}_0$, we can in fact take $\varphi' = \varphi''$ in (37). Therefore in this situation there exists a homomorphism $\varphi = \text{Ad}(v)\varphi'$ such that

$$(38) \quad \begin{aligned} \varphi(h) &= 2vc_L \\ \varphi(e) &\in {}^L\mathfrak{n} \cap \text{Ad}(w^{-1})({}^L\bar{\mathfrak{n}}). \end{aligned}$$

In view of equation (38), the roots occurring in ${}^L\mathfrak{n} \cap \text{Ad}(w^{-1})({}^L\bar{\mathfrak{n}})$ are those in $\Phi^+ \cap w^{-1}\Phi^-$. Let us denote this set of roots by

$$\Phi(w) := \Phi^+ \cap w^{-1}\Phi^-.$$

Lemma 2.17. *Let $L \in \mathcal{L}$ such that $c_L \in \overline{V}_+$ and $v, w \in W$ such that $R_{\phi, w}|_{vL_{\mathbb{C}}}$ is nonzero. Consider the restriction of the product $\frac{r(\lambda)}{r(w\lambda)}$ to vL . The roots $\alpha \in \Phi(w)$ such that the restriction of $\rho(\alpha^\vee(\lambda) + 1)$ to vL is nonconstant and appears in the denominator of the restriction of (15) to vL satisfy $\alpha^\vee(vc_L) \geq 0$.*

Proof. When we restrict $\frac{r(\lambda)}{r(w\lambda)}$ to vL certain cancellations will occur in (15) because of the additional affine integral relations between the restricted roots of the set $\Phi(w)$ due to root strings of coroots which have a constant value on vL . By (38) there exists a homomorphism $\varphi : \mathfrak{sl}_2 \rightarrow {}^L\mathfrak{g}$ such that $\varphi(h) = 2vc_L$ and $\varphi(e) \in {}^L\mathfrak{n} \cap \text{Ad}(w^{-1})({}^L\bar{\mathfrak{n}})$. Recall that φ was given by $\varphi = \text{Ad}(v)\varphi'$ with φ' satisfying $\varphi'(h) = 2c_L$ and (37). In particular, one has that

$$(39) \quad \varphi(e) \in \sum {}^L\mathfrak{g}_{\alpha^\vee},$$

with the sum over the roots $\{\alpha \in \Phi_L \mid \alpha^\vee(c_L) = 1\}$. That said, recall from \mathfrak{sl}_2 representation theory that elements in the kernel of $\text{ad}(\varphi(e))$ must be a linear combination of highest weight vectors. Thus, $\text{ad}(\varphi(e)) \in \text{End}({}^L\mathfrak{n} \cap \text{Ad}(w^{-1})({}^L\bar{\mathfrak{n}}))$ will be *injective* if restricted to a direct sum of root spaces ${}^L\mathfrak{g}_{\alpha^\vee}$ with $\alpha \in \Phi(w)$ and $\alpha^\vee(vc_L) < 0$. Now, given an $\alpha \in \Phi(w)$, let

$$\Phi(\alpha, v) := \{\beta \in \Phi(w) \mid \beta^\vee|_{vL} = \alpha^\vee|_{vL}\},$$

and similar for $\Phi(\alpha + 1, v)$. Note that if $\gamma \in \Phi_L$ with $\gamma^\vee(c_L) = 1$ and $\beta \in \Phi(\alpha, v)$, then $(\beta^\vee + v\gamma^\vee)|_{vL} = \alpha^\vee|_{vL} + 1$. Hence, from (39), it follows that

$$(40) \quad \text{ad}(\varphi(e)) : \left(\sum_{\beta \in \Phi(\alpha, v)} {}^L\mathfrak{g}_{\beta^\vee} \right) \rightarrow \left(\sum_{\beta \in \Phi(\alpha+1, v)} {}^L\mathfrak{g}_{\beta^\vee} \right).$$

Assuming further that $\alpha^\vee(vc_L) < 0$, we have that the map in (40) is injective and hence $|\Phi(\alpha, v)| \leq |\Phi(\alpha + 1, v)|$, so that, if $\Phi^{\text{comp}} = \Phi(w) \setminus (\Phi(\alpha, v) \cup \Phi(\alpha + 1, v))$,

one rewrites (15) as

$$\frac{r(\lambda)}{r(w\lambda)} = \prod_{\beta \in \Phi(\alpha, v)} \frac{\rho(\beta^\vee(\lambda))}{\rho(\beta^\vee(\lambda) + 1)} \times \prod_{\beta \in \Phi(\alpha+1, v)} \frac{\rho(\beta^\vee(\lambda))}{\rho(\beta^\vee(\lambda) + 1)} \times \prod_{\beta \in \Phi^{\text{comp}}} \frac{\rho(\beta^\vee(\lambda))}{\rho(\beta^\vee(\lambda) + 1)}$$

and concludes that $\rho(\alpha^\vee|_{vL} + 1)$ does not occur in the denominator of (15). It follows that all factors $\rho(\alpha^\vee|_{vL} + 1)$ which have a negative multiplicity (i.e. appear in the denominator of (15)) satisfy $\alpha^\vee(c_{vL}) \geq 0$. \square

With Lemma 2.17 at our disposal we are ready to describe an algorithm that consists of linear shifts of the contours of integration to compute the local residue contributions of X_{V, p_V} in a way that we will not meet the poles of R_ϕ .

We start by remarking that given a subspace $L \in \mathcal{L}(\omega)$ and a point $p_L \in L$, we will denote by $X_{L, p_L}(f)$ the integral of a function f with Schwartz decay in the imaginary direction with contour $p_L + iV^L$:

$$(41) \quad X_{L, p_L}(f) := \int_{p_L + iV^L} f(\lambda) \omega^L(\lambda),$$

in which $\omega^L(\lambda)$ is a meromorphic form on $L_{\mathbb{C}}$, regular outside the union of all codimension 1 quasi-residual subspaces of L . Such a form ω^L is produced by an iterative contour shift, as the one that we will describe.

Next, note that because the collections $\mathcal{H}(\omega)$ and $\mathcal{L}(\omega)$ are finite, there exists an $\epsilon > 0$ such that, for all $L \in \mathcal{L}(\omega)$, the ball $B_{\mathbb{R}}(c_L, \epsilon) \subseteq V$, avoids all hyperplanes of $\mathcal{H}(\omega)$ except those that contain c_L and we are thus in the situation of [O2, Proposition 3.6].

Now, a linear shift of contour from p_L to c_L should be interpreted in the following sense: in the straight line $[p_L, c_L]$, choose $\epsilon_L \in [p_L, c_L] \cap B_{\mathbb{R}}(c_L, \epsilon)$. If necessary, we deform ϵ_L in a small neighborhood to ensure that the path $[p_L, \epsilon_L]$ intersects the codimension 1 subspaces $\mathcal{L}(\omega) \ni M \subseteq L$ only in generic points p_M . Then, the linear shift from p_L to c_L is actually the shift from p_L to ϵ_L , along the line segment $[p_L, \epsilon_L]$, but we remark that the choice of ϵ_L can be made so that if $\alpha^\vee(p_L) > 0$ and $\alpha^\vee(c_L) \geq 0$ for a root $\alpha \in \Phi^+$, then, it still holds $\alpha^\vee(\epsilon_L) > 0$ as if ϵ_L was in the interior of the segment $[p_L, c_L]$. In this situation for any meromorphic function f in an open neighborhood of the form $U_L + iV$ of $c_L + iV$ which has Schwartz decay in the imaginary direction and whose poles do not meet the contours of integration in $L_{\mathbb{C}}$ of the form $p + iV^L$ with $p \in [c_L, \epsilon_L]$ we have that $X_{L, \epsilon_L}(f) = X_L^{\epsilon_L}(f(c_L + \cdot))$ for a unique tempered distribution $X_L^{\epsilon_L}$ on iV with support on iV^L . Namely it is a boundary value distribution in the sense of [Hor, Theorem 3.1.15]. These distributions $\{X_L^{\epsilon_L} \mid L \in \mathcal{L}(\omega)\}$, are the distributions X_L of (30) (see [O2], (3.28) and Section 3.1), and the superscript emphasizes that their definition depended on the choices of ϵ_L made.

Let us now describe the promised algorithm to compute the X_L . Fix $w \in W$. When we shift the contour, it could be the case that $R_{\phi, w}$ is constantly a pole for an $L \in \mathcal{L}(\omega)$ (we can only guarantee that $R_{\phi, w}$ is not constantly a pole if L is residual, see Lemma 2.15) so we may not, a priori, restrict ourselves only to the residual subspaces when performing the algorithm. For any $L \in \mathcal{L}(\omega)$ let

$$(42) \quad S(w, L) := \left\{ \alpha \in \Phi(w) \mid \begin{array}{l} \rho(\alpha^\vee|_{L_{\mathbb{C}}} + 1) \text{ is not constant and occurs} \\ \text{in the denominator of } r(\lambda)/r(w\lambda)|_{L_{\mathbb{C}}} \end{array} \right\}.$$

Inductively on k , the codimension of an affine subspace, we construct sets

- (43) $\mathcal{S}(k) := \{L \in \mathcal{L}(\omega) \mid \text{codim}(L) = k \text{ and that } L \text{ satisfies (a) - (d) below}\}$
- (a) there is a finite set $P(L) = \{p_{L,i}\} \subseteq L$ obtained by previous iterations of the algorithm,
 - (b) $\alpha^\vee(p_{L,i}) > 0$ for all $p_{L,i} \in P(L)$ and $\alpha \in S(w, L)$,
 - (c) $R_{\phi,w}|_{L_{\mathbb{C}}} \not\equiv 0$,
 - (d) $X_{V,p_V}(R_{\phi,w}) = \sum_N X_N(R_{\phi,w}) + \sum_{L,i} C(L,i)X_{L,p_{L,i}}(R_{\phi,w})$,

in which, in (d), the first sum is indexed by $\{N \in \mathcal{L} \mid \text{codim } N < k \text{ and } X_N \neq 0\}$, the second sum is indexed by $L \in \mathcal{S}(k)$ and the points $\{p_{L,i}\} \subseteq L$, and $C(L,i)$ are constants that depend on $p_{L,i}$. The construction goes as follows. Starting with $k = 0$, let

$$\mathcal{S}(0) = \{V\}, \quad P(V) = \{p_V\},$$

so that (a) - (d) are trivially satisfied. Assuming that $\mathcal{S}(k)$ is defined, perform the following steps:

- (1) For each $L \in \mathcal{S}(k)$, choose a $p_{L,0} \in P(L)$,
- (2) shift the contours of $C(L,i)X_{L,p_{L,i}}$ to $p_{L,0}$, linearly,
- (3) if $X_L \neq 0$, shift $p_{L,0}$ to c_L linearly.

Define $\mathcal{S}(k+1)$ to be the collection of $M \in \mathcal{L}(\omega)$ of codimension 1 in each $L \in \mathcal{S}(k)$ that were crossed on points p_M during the linear shifts of contour described in steps (2) and (3) and such that $R_{\phi,w}|_M \not\equiv 0$.

Lemma 2.18. *In both shifts of contour described in (2) and (3), the pole-set of $\frac{r(\lambda)}{r(w\lambda)}|_{L_{\mathbb{C}}}$ is not crossed.*

Proof. Recall that the pole-set of $\frac{r(\lambda)}{r(w\lambda)}$ is a union of hyperplanes $\alpha^\vee(\lambda) = z$, with $\text{Re}(z) \in (-1, 0)$. Thus, to prove the Lemma it suffices to show that for every point p involved in the linear shifts described in steps (2) and (3) we have $\alpha^\vee(p) \geq 0$ for all $\alpha \in S(w, L)$.

In step (2), because of property (b) of the set $\mathcal{S}(k)$, any point $p \in [p_{L,i}, p_{L,0}]$ satisfy $\alpha^\vee(p) > 0$ for all $\alpha \in S(w, L)$. Further, the condition that $X_L \neq 0$ to perform step (3) implies that L is residual (see Remark 2.8), so if we write $L = vN$ with N residual in dominant position and $v \in W$ we have $c_L = vc_N$ and we are in condition to use Lemma 2.17 to conclude that for all p in the interior of the segment $[p_{L,0}, c_L]$ it holds that $\alpha^\vee(p) > 0$ for every $\alpha \in S(w, L)$. \square

Proposition 2.19. *The set $\mathcal{S}(k+1)$ satisfies (a) - (d), so that the induction is well-defined.*

Proof. When we perform the linear shifts as in (2) and (3), we cross subspaces $M \in \mathcal{L}(\omega)$ of codimension 1 in the $L \in \mathcal{S}(k)$. Notice that a same codimension 1 subspace $M \in \mathcal{L}(\omega)$ may be crossed several times during the shift of different L , but as all the collection of hyperplanes $\mathcal{L}(\omega)$ is finite, we end up with a finite set $P(M)$ as in (a). These possible multiple crossings of subspace M is what creates the constants $C(i)$. Moreover, in the proof of Lemma 2.18, we saw that all points p_M produced satisfy $\alpha^\vee(p_M) > 0$, for all $\alpha \in S(w, L)$. Now, note that if $\beta \in S(M, w)$, and $M \subseteq L$, then there exist $\alpha \in S(L, w)$ such that $\alpha^\vee|_M = \beta^\vee|_M$. Hence, (b) is also satisfied and (c) is satisfied by construction. We are left to show that condition (d) holds. By the inductive hypothesis, it holds for $\mathcal{S}(k)$. Next, note that when we

perform step (2), we combine all the constants $C(i)$ to obtain a unique integration with contour $p_{L,0} + iV^L$ and we get

$$\sum_{L \in \mathcal{S}(k)} \sum_i C(i) X_{L,p_{L,i}}(R_{\phi,w}) = \sum_{L \in \mathcal{S}(k)} X_{L,p_{L,0}}(R_{\phi,w}) + \{\text{codimension 1 integrals}\}.$$

Because of property (b), we have that $R_{\phi,w}|_{L_{\mathbb{C}}}$ is not constantly a pole along L . Thus, if it is the case that $X_L = 0$, from [Hor, Theorem 3.1.15], one has that actually the sum of the coefficient functions used to define X_L (see [O2, (3.28)]) are already zero, and hence, the integration $X_{L,p_{L,0}}(R_{\phi,w}) = 0$. Hence, we can disconsider every $L \in \mathcal{S}(k)$ such that $X_L = 0$ and only the X_L with L residual remain. Finally, when we shift the contour using step (3), we obtain a contribution $X_L(R_{\phi,w})$ plus some codimension 1 integrals for $M \in \mathcal{S}(k+1)$, say $X_{M,p_M}(R_{\phi,w})$. We thus obtain

$$X_{V,p_V}(R_{\phi,w}) = \sum_{\{N \in \mathcal{L} \mid \text{codim } N \leq k\}} X_N(R_{\phi,w}) + \sum_{M,i} C(i) X_{M,p_{M,i}}(R_{\phi,w}),$$

with the last sum ranging over all the $M \in \mathcal{S}(k+1)$ and the $p_{M,i}$ just constructed, so (d) is satisfied. \square

Corollary 2.20. *The inner product $(\theta_{\phi}, \theta_{\psi})$ decomposes as*

$$(44) \quad (\theta_{\phi}, \theta_{\psi}) = \sum_{c \in \mathcal{C}} X_c(R_{\phi}\psi(c + \cdot)).$$

Proof. After the induction procedure described above we obtain $X_{V,p_V}(R_{\phi}\psi) = \sum_L X_L(R_{\phi}\psi)$, with the sum over all the L such that $X_L \neq 0$ and such that there is a $w \in W$ for which $R_{\phi,w}|_{L_{\mathbb{C}}} \neq 0$. Grouping all the terms and using (29), (30) and (20), the result follows. \square

2.5. The expression for the scalar product. The last task is to rewrite (44) in a symmetric form to obtain a useful expression. Using (28) and a computation similar to [HO1, Theorem 3.18], we obtain:

$$(45) \quad (\theta_{\phi}, \theta_{\psi}) = |W|^{-1} \sum_{L \in \mathcal{L}} \int_{L^t} \sum_{v,w \in W} \phi^{-}(-w\lambda) \psi(v\lambda) c_{\mathbb{H}}(-w\lambda) c_{\mathbb{H}}(v\lambda) \frac{r(v\lambda)}{r(w\lambda)} d\nu_L(\lambda),$$

in which ν_L is the unique positive measure supported on $L^t = c_L + iV^L \subseteq V_{\mathbb{C}}$ characterized by the requirement that, for all $f \in PW(V_{\mathbb{C}})$, we have

$$\int_{L^t} f d\nu_L = Y_L(f(c_L + \cdot))$$

if c_L is dominant, and such that

$$(46) \quad \nu_{WL} := \sum_{L' \subseteq WL} \nu_{L'}$$

is a W -invariant measure. Moreover, ν_L is the push forward of a smooth measure on $c_L + iV^L$ and is of the form (see [HO1, Definition 3.17] and Theorem 2.4): for $\lambda = c_L + i\lambda^L \in L^t$,

$$(47) \quad d\nu_L(\lambda) := \frac{|W(\Phi_L)_{c_L}|}{|A_{\mathbf{o}_{WL}}|} \frac{\prod'_{\alpha \in \Phi_L} \alpha^{\vee}(c_L)}{\prod'_{\alpha \in \Phi_L} (\alpha^{\vee}(c_L) + 1)} \prod_{\alpha \in \Phi^+ \setminus \Phi_L} \frac{c_L(\alpha^{\vee})^2 + \alpha^{\vee}(\lambda^L)^2}{(c_L(\alpha^{\vee}) - 1)^2 + \alpha^{\vee}(\lambda^L)^2} d\lambda^L,$$

where $|A_{\mathfrak{o}_{WL}}|$ is the cardinality of the component group of the centralizer of the image of the \mathfrak{sl}_2 -homomorphism associated to WL (see Proposition 2.10) and \prod' denotes the product over the nonzero factors. We remark that the precise constants in (47) were not yet available in [HO1], but these can be derived by a limit procedure from the explicit formal degree formulas for discrete series characters of Iwahori-Hecke algebras [Re2, (0.3)] and [CKK]. These explicit formulas constitute a special case of the proof of the conjecture [HII, Section 3.4] for unipotent representations of semisimple groups of adjoint type over a non-Archimedean local field [O3, Section 4.6]. The limit procedure and the computation of the relevant constants in the present case of graded Hecke algebras will appear elsewhere [DMO].

Let us rewrite (45) in order to exhibit its most important properties, namely that in this formula every summand corresponding to a W -orbit of residual subspaces contributes a positive semidefinite Hermitian form on the space of Paley-Wiener functions. For that, let $W \setminus \mathcal{L}$ denote a complete set of representatives for W -orbits of residual subspaces. Note that if $\lambda = c_L + i\mu \in L^t$ and if $w_L \in W(\Phi_L)$ denotes the longest element in the Weyl group $W(\Phi_L)$ then $w_L(c_L) = -c_L$ and $w_L\mu = \mu$. In other words, for $\lambda \in L^t$ we have:

$$(48) \quad -w_L\lambda = \bar{\lambda}.$$

Lemma 2.21. *The expression (45) for the inner product can be written in the following equivalent way:*

$$(49) \quad \sum_{L \in W \setminus \mathcal{L}} |W| \int_{WL^t} \overline{A_0(r\phi)(\lambda)} A_0(r\psi)(\lambda) \frac{d\nu_{WL}(\lambda)}{r(-\lambda)r(\lambda)}.$$

Proof. First observe that $r(-\lambda)r(\lambda)$ does not vanish identically on L^t . Indeed, $\alpha \in \Phi$ is constant on L^t if and only if $\alpha \in \Phi_L$. By Proposition 2.10(d), the constant value of $r(-\lambda)r(\lambda)$ on L^t is in \mathbb{Z} , hence outside of the critical strip. Thus it makes sense to multiply and divide the integrand of (45) by $r(-w\lambda)$, for each $w \in W$, to obtain

$$(r(-\lambda)r(\lambda))^{-1} \sum_{v, w \in W} \phi^-(w\lambda) c_{\mathbb{H}}(-w\lambda) r(-w\lambda) \psi(v\lambda) c_{\mathbb{H}}(v\lambda) r(v\lambda).$$

Then, using (48) and the definition of A_0 in (26), one rewrites this as

$$|W|^2 (r(-\lambda)r(\lambda))^{-1} \overline{A_0(r\phi)(\lambda)} A_0(r\psi)(\lambda),$$

and the result follows from the equality $A_0(r\phi^-)(-\lambda) = \overline{A_0(r\phi)(\lambda)}$, on L^t . \square

Lemma 2.22. *The integrand of each factor of (49) is Hermitian, regular and non-negative on L^t .*

Proof. As ρ takes positive values on \mathbb{Z} , the constant factors of r on L^t are all positive. This implies that $r(-\lambda)r(\lambda)$ is a nonnegative function on L^t . Indeed, using that $-w_L$ is a permutation of $\Phi \setminus \Phi_L$ we have

$$r(-\lambda)r(\lambda) = R \prod_{\alpha \in \Phi^+ \setminus \Phi_L} \rho(\alpha^\vee(\lambda)) \overline{\rho(\alpha^\vee(\lambda))},$$

in which use was made of (48) and $R > 0$ is the product of the constant values of ρ . Thus, the factor $\overline{A_0(r\phi)(\lambda)} A_0(r\psi)(\lambda)$ is entire on $V_{\mathbb{C}}$, W -invariant, and nonnegative on L^t if $\phi = \psi$. Interchanging the role of ϕ and ψ clearly results in complex conjugation of the restriction to L^t . On the other hand, $(r(-\lambda)r(\lambda))^{-1}$ is meromorphic on

$V_{\mathbb{C}}$, W -invariant, and nonnegative on L^t . However, *it may possibly have singularities on L^t corresponding to critical zeros of ρ* . But the product

$$\overline{A_0(r\phi)(-\lambda)} A_0(r\psi)(\lambda) (r(-\lambda)r(\lambda))^{-1},$$

which can be written as (45) by Lemma 2.21, is the restriction to L^t of a meromorphic function which has no singularities on L^t or is identically zero. \square

Let us now consider the growth behavior on L^t of each factor of (49). Lemma 2.17 implies also that each summand of the integrand written in the form (45) on L^t is given by the product of certain Paley-Wiener functions on L^t (we can, because of the double summation over W , absorb the rational factors $c_{\mathbb{H}}$ by these Paley-Wiener functions) times factors of the form $\frac{\rho(\alpha^\vee(\lambda)+a)}{\rho(\alpha^\vee(\lambda)+b)}$ where a and b are in \mathbb{R} such that the argument is *not* in the critical strip. Such factors are of moderate growth on L^t by the analytical properties of $\rho(s)$ and discussed in subsection 2.1. We have, therefore, shown that:

Theorem 2.23. *For each $L \in W \setminus \mathcal{L}$, let ν_{WL} be as in (46) and μ_{WL} be the positive measure on WL^t defined by*

$$(50) \quad d\mu_{WL}(\lambda) := \left(\frac{|W|}{r(-\lambda)r(\lambda)} \right) d\nu_{WL}(\lambda).$$

Then, the corresponding summand of (49)

$$\langle \phi, \psi \rangle_{WL} := \int_{WL^t} \overline{A_0(r\phi)(\lambda)} A_0(r\psi)(\lambda) d\mu_{WL}(\lambda)$$

defines a positive semidefinite Hermitian form on the space of Paley-Wiener functions on $V_{\mathbb{C}}$. The radical of this pairing consists of Paley-Wiener functions ϕ for which we have $A_0(r\phi)|_{WL^t} = 0$. We have a continuous map, isometric with respect to $\langle \phi, \psi \rangle_{WL}$ and with dense image $A_{WL} : \text{PW}(V_{\mathbb{C}}) \rightarrow L^2(WL^t, \mu_{WL})^W$ given by $\phi \mapsto A_0(r\phi)|_{WL^t}$. Finally we have

$$(\theta_\phi, \theta_\psi) = \sum_{L \in W \setminus \mathcal{L}} \langle \phi, \psi \rangle_{WL}.$$

In view of Proposition 2.10 and the bijection between unipotent classes and nilpotent orbits, we can choose the representatives of $W \setminus \mathcal{L}$ used in Theorem 2.23 to be consistent with the choice of representatives for the ${}^L G$ -classes of pairs $({}^L M, {}^L P)$ we made in the Introduction, so that we have bijections

$$(51) \quad \mathbf{o} \leftrightarrow L_{\mathbf{o}} \leftrightarrow ({}^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}}).$$

We set $\mu_{\mathbf{o}} := \mu_{WL_{\mathbf{o}}}$ and $\langle \cdot, \cdot \rangle_{\mathbf{o}} := \langle \cdot, \cdot \rangle_{WL_{\mathbf{o}}}$ and we obtain Theorem 1.

2.6. Weyl groups. We conclude this section by noting that, in Theorem 2.23, instead of considering W -invariant functions on the orbit WL^t , we may concentrate only on a fixed representative of the orbit and consider $W(L)$ -invariant functions on L^t , where $W(L) = \text{Stab}(L)/\text{Fix}(L)$ is the Weyl group of the affine subspace L . We scale the measure $\tilde{\mu}_L = (|W|/|\text{Stab}(L)|)\mu_{WL}|_{L^t}$ so that we can write

$$(52) \quad L^2(WL^t, \mu_{WL})^W = L^2(L^t, \tilde{\mu}_L)^{W(L)}.$$

Moreover, in view of the bijections discussed in (51), we can identify the Weyl group $W(L_{\mathbf{o}})$ with the Weyl group of the pair $({}^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}})$

$$(53) \quad W({}^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}}) := \{w \in N_{L_G}({}^L M_{\mathbf{o}}) / {}^L M_{\mathbf{o}} \mid w({}^L P_{\mathbf{o}})w^{-1} \text{ is conjugate to } {}^L P_{\mathbf{o}} \text{ in } {}^L M_{\mathbf{o}}\}.$$

We may write, accordingly, that the conclusion of Theorem 1 is that

$$(54) \quad L^2(G(F)Z_G \backslash G(\mathbb{A}))_{[T,1]}^{\mathbf{K}} \cong \oplus_{\mathbf{o}} L^2(L_{\mathbf{o}}^t, \tilde{\mu}_{\mathbf{o}})^{W({}^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}})}.$$

3. NORMALIZED EISENSTEIN SERIES

3.1. Proof of Theorem 2. Recall that the unramified Borel Eisenstein series is given by

$$(55) \quad E(\lambda, g) = \sum_{\gamma \in B(F) \backslash G(F)} t_B(\gamma g)^{\lambda + \varrho},$$

where t_B is the map coming from the Iwasawa decomposition and $\varrho \in a_T^*$ is the Weyl vector. It is well known that this series is absolutely convergent for $\operatorname{Re}(\lambda) = \lambda_0 \gg 0$, has meromorphic continuation for all $\lambda \in a_{T,\mathbb{C}}^{G*}$ and satisfies functional equations

$$(56) \quad E(w\lambda, g) = \frac{c(w\lambda)}{c(\lambda)} E(\lambda, g),$$

for all $w \in W$. For $g \in G(\mathbb{A})$, we define the **normalized Eisenstein series** by

$$(57) \quad E_0(\lambda, g) := \frac{1}{|W|} A_0(rE(-\cdot, g))(-\lambda).$$

It follows from Theorem 2 that $E_0(\lambda(\mathbf{o}), g)$ is a square-integrable and \mathbf{K} -invariant function on $G(F)Z_G \backslash G(\mathbb{A})$ for each distinguished unipotent orbit \mathbf{o} .

Lemma 3.1. *The normalized Eisenstein $E_0(\lambda, g)$ is, as a function of λ , holomorphic, W -invariant and satisfies*

$$E_0(\lambda, g) = \frac{1}{|W|} c_{\mathbb{H}}(-\lambda) r(-\lambda) E(\lambda, g).$$

Proof. The formula is a straightforward computation using A_0 . Together with the fact implied by the functional equations (56) that $E(\lambda, g)$ has zeroes along the hyperplanes $\alpha^\vee(\lambda) = 0$ for α simple, it follows that $E_0(\lambda, g)$ is holomorphic in the closure of the fundamental chamber. Invoking the W -invariance implied by the averaging operator, it is holomorphic everywhere. \square

Lemma 3.2. *Given $\phi \in PW(a_{T,\mathbb{C}}^{G*})$ define*

$$u_\phi(\lambda) := \int_{G(F)Z_G \backslash G(\mathbb{A})} \overline{\theta_\phi(g)} E(\lambda, g) dg$$

This integral converges for $\operatorname{Re}(\lambda) = \lambda_0 \gg 0$ and defines a holomorphic function in its domain of convergence that satisfies

$$u_\phi(\lambda) = c_{\mathbb{H}}(-\lambda)^{-1} R_\phi(\lambda).$$

Proof. Given $\psi \in PW(a_{T,\mathbb{C}}^{G*})$, one has

$$\int_{\operatorname{Re}(\lambda)=\lambda_0 \gg 0} \psi(\lambda) \int_{G(F)Z_G \backslash G(\mathbb{A})} \overline{\theta_\phi(g)} E(\lambda, g) dg \, d\lambda = \int_{\operatorname{Re}(\lambda)=\lambda_0 \gg 0} R_\phi(\lambda) \psi(\lambda) \frac{d\lambda}{c_{\mathbb{H}}(-\lambda)},$$

in which the exchange of integrals is allowed by the estimates used in the proof of [MW2, Proposition II.1.10] and use was made of equations (2) and (16). Since this holds for all $\psi \in PW(a_{T,\mathbb{C}}^{G*})$ the result follows. \square

Lemma 3.3. *Given $\phi \in PW(a_{T,\mathbb{C}}^{G*})$, it holds that*

$$\int_{G(F)Z_G \backslash G(\mathbb{A})} \overline{\theta_\phi(g)} E_0(\lambda, g) dg = A_0(r\phi^-)(-\lambda).$$

Proof. Using Lemmas 3.1, 3.2 and (17), we get

$$\begin{aligned} \int_{G(F)Z_G \backslash G(\mathbb{A})} \overline{\theta_\phi(g)} E_0(\lambda, g) dg &= \frac{1}{|W|} c_{\mathbb{H}}(-\lambda) r(-\lambda) u_\phi(\lambda) \\ &= \frac{1}{|W|} \sum_{w \in W} c_{\mathbb{H}}(-w\lambda) r(-w\lambda) \phi^-(-w\lambda), \end{aligned}$$

proving the Lemma. \square

Using that $-\bar{\lambda} \in W\lambda$ so that $A_0(r\phi^-)(-\lambda) = \overline{A_0(r\phi)(\lambda)}$ and $\overline{E_0(\lambda, g)} = E_0(-\lambda, g)$ for all λ in the support $\cup_{\mathbf{o}} WL_{\mathbf{o}}^t$, it follows that $\mathcal{F}(\theta_\phi)(\lambda) = A_0(r\phi)(\lambda)$ and hence the isometry part of Theorem 2 follows immediately from Theorem 1.

For v non-Archimedean, $\mathcal{H}(G_v, K_v)$ is the algebra of compactly supported functions on $K_v \backslash G_v / K_v$. We let $S_v : \mathcal{H}(G_v, K_v) \rightarrow \mathbb{C}[X_*(T)]^W$ denote the Satake isomorphism and the $*$ -structure of $\mathcal{H}(G_v, K_v)$ can be described via

$$(58) \quad S_v(h^*)(t) = \overline{S_v(h)(t^{-1})}.$$

For v Archimedean $\mathcal{H}(G_v, K_v)$ is the subalgebra of left and right invariant elements of the corresponding Archimedean Hecke algebra (the Archimedean Hecke algebra is described in [F, Paragraph 3] or [BJ, 1.1] although our notation differs from the one in [BJ]). We let $S_v : \mathcal{H}(G_v, K_v) \rightarrow \text{Sym}[\mathbb{C} \otimes X_*(T)]^W$ denote now the Harish-Chandra isomorphism and one can describe the $*$ -structure similarly as

$$(59) \quad S_v(h^*)(\lambda) = \overline{S_v(h)(-\bar{\lambda})}.$$

Each $\lambda \in a_{T,\mathbb{C}}^{G*}$ defines a character $\chi_{v,\lambda}$ of $\mathcal{H}(G_v, K_v)$ by means of S_v and thus a character χ_λ of $\mathcal{H}(G(\mathbb{A}), \mathbf{K})$. A straightforward computation yields that, if λ is such that $-\bar{\lambda} \in W\lambda$ (see 48), then

$$(60) \quad \chi_\lambda(h^*) = \overline{\chi_\lambda(h)},$$

and it is known that the unramified Borel Eisenstein series $E(\lambda, g)$ is an eigenfunction for the convolution action of $\mathcal{H}(G_v, K_v)$ with eigenvalue $\chi_{v,\lambda}$ for each v . It then follows that, for $h_v \in \mathcal{H}(G_v, K_v)$ and for $\lambda \in \cup_{\mathbf{o}} WL_{\mathbf{o}}^t$, we have

$$\begin{aligned} \mathcal{F}(h_v \cdot f)(\lambda) &= \int_{G(F)Z_G \backslash G(\mathbb{A})} E_0(-\lambda, g) (h_v \cdot f)(g) dg \\ &= \int_{G(F)Z_G \backslash G(\mathbb{A})} \overline{(h_v^* \cdot E_0(\lambda, g))} f(g) dg \\ &= \chi_{v,\lambda}(h_v) \mathcal{F}(f)(\lambda), \end{aligned}$$

proving the equivariance of \mathcal{F} .

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